Duality Between Some Linear Preserver Problems. II. Isometries with Respect to c-Spectral Norms and Matrices with Fixed Singular Values

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ABSTRACT

Let $\mathbb{F}_{m\times n}$ ($m \leqslant n$) denote the linear space of all $m\times n$ complex or real matrices according as $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Let $c = (c_1,\dots,c_m) \neq 0$ be such that $c_1 \geqslant \cdots \geqslant c_m \geqslant 0$. The c-spectral norm of a matrix $A \in \mathbb{F}_{m\times n}$ is the quantity

$$\|A\|_c = \sum_{i=1}^{m} c_i \sigma_i(A),$$

where $\sigma_1(A) \geqslant \cdots \geqslant \sigma_m(A)$ are the singular values of A. Let $d = (d_1,\dots,d_m) \neq 0$, where $d_1 \geqslant \cdots \geqslant d_m \geqslant 0$. We consider the linear isometries between the normed spaces $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$ and $(\mathbb{F}_{m\times n}, \|\cdot\|_d)$, and prove that they are dual transformations of the linear operators which map $\mathcal{S}(d)$ onto $\mathcal{S}(c)$, where

$$\mathcal{S}(c) = \{X \in \mathbb{F}_{m\times n} : X \text{ has singular values } c_1,\dots,c_m\}.$$

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It is shown that such isometries (and hence their dual transformations) exist if and only if c and d are scalar multiples of each other. In that case, we completely determine the structure of such isometries, and prove that they and their dual transformations belong to a same class of operators. In the proof, we obtain a characterization of the extreme points of the unit ball in $\mathbb{F}_{m\times n}$ with respect to $\|\cdot\|_c$, which is of independent interest.

1. INTRODUCTION

Let $\mathbb{F}_{m\times n}$ denote the set of all $m\times n$ complex or real matrices according as $\mathbb{F}=\mathbb{C}$ or $\mathbb{F}=\mathbb{R}$. We assume $m\leqslant n$ without loss of generality. Denote by A^* the conjugate transpose or transpose of $A\in\mathbb{F}_{m\times n}$ according as $\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$, and by

$$\mathscr{U}_k=\{A\in\mathbb{F}_{k\times k}: AA^*=I\}$$

the unitary (orthogonal) group in $\mathbb{F}_{k\times k}$. For any $A\in\mathbb{F}_{m\times n}$, its *singular values* $\sigma_1(A)\geqslant\sigma_2(A)\geqslant\cdots\geqslant\sigma_m(A)\geqslant 0$ are the nonnegative square roots of the eigenvalues of the matrix AA^*. Suppose $c=(c_1,\dots,c_m)\neq 0$ is given such that $c_1\geqslant\cdots\geqslant c_m\geqslant 0$. Then the *c-spectral norm* of A (see [12]) is the quantity

$$\|A\|_c=\sum_{i=1}^{m}c_i\sigma_i(A).$$

It is not hard to verify that $\|\cdot\|_c$ is a unitarily invariant norm (see [7]) on $\mathbb{F}_{m\times n}$, i.e., $\|\cdot\|_c$ is a norm such that

$$\|A\|_c=\|UAV\|_c \qquad \text{for all} \quad A\in\mathbb{F}_{m\times n},\quad U\in\mathscr{U}_m,\quad V\in\mathscr{U}_n.$$

When $c_1=\cdots=c_k=1$, $c_{k+1}=\cdots=c_m=0$, $\|\cdot\|_c$ reduces to the Ky Fan k-norm $\|\cdot\|_k$. It is known (see [7, p. 445]) that the collection of Ky Fan k-norms $(1\leqslant k\leqslant m)$ forms an important subset of the set of all unitarily invariant norms on $\mathbb{F}_{m\times n}$. Several authors (see [6, 5, 8]) have studied the linear operators T on $\mathbb{F}_{m\times n}$ that preserve the Ky Fan k-norms. With very few exceptions (see [8]), these operators T are of the form

$$T(A)=UAV \qquad \text{for all} \quad A\in\mathbb{F}_{m\times n}, \tag{1}$$

or

$$m=n \quad \text{and} \quad T(A)=UA^tV \quad \text{for all} \quad A\in\mathbb{F}_{m\times n}, \tag{2}$$

where A^t denotes the transpose of A, and U, V are some fixed elements in $\mathscr{U}_m$ and $\mathscr{U}_n$ respectively. In this note we consider a more general problem. For any nonzero $c=(c_1,\dots,c_m)$ and $d=(d_1,\dots,d_m)$, where $c_1 \geqslant \cdots \geqslant c_m \geqslant 0$ and $d_1 \geqslant \cdots \geqslant d_m \geqslant 0$, we consider those linear isometries T from the normed space $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$ to the normed space $(\mathbb{F}_{m\times n}, \|\cdot\|_d)$, i.e., those linear operators $T: \mathbb{F}_{m\times n} \to \mathbb{F}_{m\times n}$ which satisfy

$$\|T(A)\|_d = \|A\|_c \qquad \text{for all} \quad A \in \mathbb{F}_{m\times n}.$$

We prove in Section 2 that such linear isometries are dual transformations of those linear operators which map $\mathscr{S}(d)$ onto $\mathscr{S}(c)$, where $\mathscr{S}(c)$ is defined as

$$\begin{aligned}\mathscr{S}(c) &= \{X \in \mathbb{F}_{m\times n} : X \text{ has singular values } c_1,\dots,c_m\} \\ &= \left\{U\left(\sum_{i=1}^{m} c_i E_{ii}\right)V : U \in \mathscr{U}_m,\ V \in \mathscr{U}_n\right\},\end{aligned}$$

and $\{E_{11}, E_{12},\dots, E_{mn}\}$ is the standard basis for $\mathbb{F}_{m\times n}$. In Section 3, we give a characterization of the extreme points of the unit ball in $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$. With this, we prove that linear isometries from $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{F}_{m\times n}, \|\cdot\|_d)$ (and hence their dual transformations) exist if and only if c and d are scalar multiples of each other. When c is a scalar multiple of d, we determine the structure of such linear isometries and their dual transformations, and show that they actually belong to a same class of operators, which, with only a few exceptions, "basically" consists of those described in (1) and (2). The different cases of $\mathbb{F}=\mathbb{C}$ and $\mathbb{F}=\mathbb{R}$ are dealt with in Sections 4 and 5 separately. Section 6 is a variation of the problem, which considers $\mathbb{C}_{m\times n}$ as a real linear space of dimension $2mn$. Some remarks are given in Section 7.

To avoid trivialities, we shall always assume $2 \leqslant m \leqslant n$ throughout the paper. By isometries we always mean linear isometries. We define

$$\mathbb{R}_+^m\!\downarrow\; = \{(c_1,\dots,c_m) : c_1 \geqslant \cdots \geqslant c_m \geqslant 0,\ c_1 \neq 0\}$$

and always assume $c, d \in \mathbb{R}_+^m\!\downarrow$. We also remark that the sets $\mathscr{U}_k$, $\mathscr{S}(c)$, and $\mathscr{S}(d)$ depend on the situations of $\mathbb{F}=\mathbb{C}$ or $\mathbb{F}=\mathbb{R}$. We shall use the same notation for these sets under the two different situations.

2. DUALITY BETWEEN THE TWO LINEAR PRESERVER PROBLEMS

Let $\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$. In $\mathbb{F}_{m\times n}$, let $\langle\cdot,\ \cdot\rangle$ denote the usual inner product defined by

$$\langle A, B\rangle = \operatorname{tr}(AB^*).$$

For any linear operator on $\mathbb{F}_{m\times n}$, its dual transformation is the unique linear operator T^* on $\mathbb{F}_{m\times n}$ which satisfies

$$\langle T(A), B\rangle = \langle A, T^*(B)\rangle \qquad \text{for all} \quad A, B\in\mathbb{F}_{m\times n}.$$

We establish the duality between the isometries from $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{F}_{m\times n}, \|\cdot\|_d)$ and the operators which map $\mathscr{S}(d)$ onto $\mathscr{S}(c)$ in the following theorem.

THEOREM 1. *Let $c, d\in\mathbb{R}^m_+\downarrow$. A linear operator $T:\mathbb{F}_{m\times n}\to\mathbb{F}_{m\times n}$ will satisfy*

$$\|T(A)\|_d = \|A\|_c \qquad \text{for all } A$$

if and only if its dual transformation T^* satisfies

$$T^*(\mathscr{S}(d)) = \mathscr{S}(c).$$

Proof. Let $c=(c_1,\dots,c_m)$, $d=(d_1,\dots,d_m)$ be in $\mathbb{R}^m_+\downarrow$. If $A\in\mathbb{F}_{m\times n}$ and $X\in\mathscr{S}(c)$, then

$$\begin{aligned}
|\langle A, X\rangle| &= |\operatorname{tr}(AX^*)|\\
&\leqslant \sum_{i=1}^{m}\sigma_i(AX^*) \qquad \text{(see [3, p. 47])}\\
&\leqslant \sum_{i=1}^{m}\sigma_i(A)\sigma_i(X) \qquad \text{(see [3, p. 49])}\\
&= \|A\|_c.
\end{aligned}$$

On the other hand, if A has singular value decomposition UDV (see [7]) with

$U \in \mathcal{U}_m$, $V \in \mathcal{U}_n$, and $D = \sum_{i=1}^m \sigma_i(A) E_{ii}$, we may choose $X_0 = UD_0V \in \mathcal{S}(c)$ with $D_0 = \sum_{i=1}^m c_i E_{ii}$. Then

$$|\langle A, X_0 \rangle| = |\langle UDV, UD_0V \rangle| = |\langle D, D_0 \rangle| = \|A\|_c.$$

Hence

$$\|A\|_c = \max\{|\langle A, X \rangle| : X \in \mathcal{S}(c)\}.$$

As a result, if $T : \mathbb{F}_{m\times n} \to \mathbb{F}_{m\times n}$ is linear and $T^*(\mathcal{S}(d)) = \mathcal{S}(c)$, then for any $A \in \mathbb{F}_{m\times n}$,

$$\begin{aligned}
\|T(A)\|_d &= \max\left\{\left|\langle T(A), X \rangle\right| : X \in \mathcal{S}(d)\right\} \\
&= \max\left\{\left|\langle A, T^*(X) \rangle\right| : X \in \mathcal{S}(d)\right\} \\
&= \max\{|\langle A, X \rangle| : X \in \mathcal{S}(c)\} \\
&= \|A\|_c.
\end{aligned}$$

Conversely, suppose T is an isometry from $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{F}_{m\times n}, \|\cdot\|_d)$. As for any linear functional f on $\mathbb{F}_{m\times n}$, we can find $A \in \mathbb{F}_{m\times n}$ such that

$$\operatorname{Re} f(X) = \operatorname{Re}\langle A, X \rangle \qquad \text{for all} \quad X \in \mathbb{F}_{m\times n},$$

we have

$$\begin{aligned}
\max\{\operatorname{Re} f(X) : X \in \mathcal{S}(c)\} &= \max\{\operatorname{Re}\langle A, X \rangle : X \in \mathcal{S}(c)\} \\
&= \max\{|\langle A, X \rangle| : X \in \mathcal{S}(c)\} \\
&= \|A\|_c = \|T(A)\|_d \\
&= \max\left\{\left|\langle T(A), X \rangle\right| : X \in \mathcal{S}(d)\right\} \\
&= \max\left\{\left|\langle A, T^*(X) \rangle\right| : X \in \mathcal{S}(d)\right\} \\
&= \max\{|\langle A, Y \rangle| : Y \in T^*(\mathcal{S}(d))\} \\
&= \max\{\operatorname{Re} f(Y) : Y \in T^*(\mathcal{S}(d))\}.
\end{aligned}$$

By the separation theorem [15, p. 58], we get

$$T^*[\operatorname{conv}\mathscr{S}(d)] = \operatorname{conv} T^*(\mathscr{S}(d)) = \operatorname{conv}\mathscr{S}(c), \tag{3}$$

where "conv" means "the convex hull of." For any subset S of a linear space, the set of extreme points of S is

$$\operatorname{ext} S = \left\{ X \in S : X \neq \frac{P+Q}{2} \text{ for any distinct } P, Q \in S \right\}.$$

Since $\|\cdot\|_c$ and $\|\cdot\|_d$ are norms on $\mathbb{F}_{m\times n}$, the isometry T must be one-one. Hence T^* is also one-one. Then (3) implies

$$T^*[\operatorname{ext}\operatorname{conv}\mathscr{S}(d)] = \operatorname{ext}\operatorname{conv}\mathscr{S}(c).$$

As $\operatorname{ext}\operatorname{conv}\mathscr{S}(x) = \mathscr{S}(x)$ for all $x \in \mathbb{R}^m_+\downarrow$, the result follows. ∎

3. EXTREME POINTS OF THE UNIT BALL IN $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Given any $c = (c_1, \dots, c_m) \in \mathbb{R}^m_+\downarrow$, we define the unit ball in $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$ by

$$\mathscr{B}(c) = \{ A \in \mathbb{F}_{m\times n} : \|A\|_c \leqslant 1 \},$$

and the set of its extreme points by

$$\mathscr{E}(c) = \operatorname{ext}\mathscr{B}(c).$$

Moreover, for any $k = 1, \dots, m$, we set

$$\mathscr{B}_k = \{ A \in \mathbb{F}_{m\times n} : \|A\|_k \leqslant 1 \},$$

$$\mathscr{E}_k = \operatorname{ext}\mathscr{B}_k,$$

$$J_k = \sum_{i=1}^{k} E_{ii},$$

and

$$\mathscr{S}_k = \{A \in \mathbb{F}_{m\times n} : \sigma_1(A) = \cdots = \sigma_k(A) = 1,\ \sigma_{k+1}(A) = \cdots = \sigma_m(A) = 0\}$$
$$= \{UJ_kV : U \in \mathscr{U}_m,\ V \in \mathscr{U}_n\}.$$

We remark that the sets $\mathscr{B}(c)$, $\mathscr{E}(c)$, $\mathscr{B}_k$, $\mathscr{E}_k$, and $\mathscr{S}_k$ will depend on the situations of $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. However, we shall use the same notation in these two different situations.

A complete characterization of $\mathscr{E}_k$ is given in [5, Theorem 4]. We shall characterize $\mathscr{E}(c)$ for general c in Theorem 2.

LEMMA 1. *For any $c = (c_1, \dots, c_m) \in \mathbb{R}^m_+\downarrow$, we have*

$$\mathscr{E}(c) \subset \bigcup_{j=1}^{m} \left(\sum_{i=1}^{j} c_i \right)^{-1} \mathscr{S}_j.$$

Proof. Let $A \in \mathscr{E}(c)$ have singular value decomposition UDV, where $D = \sum_{i=1}^m \sigma_i(A) E_{ii}$. Then clearly

$$1 = \|A\|_c = \sum_{i=1}^{m} c_i \sigma_i(A).$$

Moreover, $\sigma(A) = (\sigma_1(A), \dots, \sigma_m(A))$ must be an extreme point of the set

$$C = \left\{ (x_1, \dots, x_m) \in \mathbb{R}^m : x_1 \geqslant \cdots \geqslant x_m \geqslant 0,\ \sum_{i=1}^{m} c_i x_i \leqslant 1 \right\}.$$

Otherwise $\sigma(A) = (p+q)/2$, where $p = (p_1, \dots, p_m)$ and $q = (q_1, \dots, q_m)$ are distinct elements in C. It follows that $A = (P+Q)/2$, where

$$P = U\left(\sum_{i=1}^{m} p_i E_{ii} \right) V \quad \text{and} \quad Q = U\left(\sum_{i=1}^{m} q_i E_{ii} \right) V$$

are distinct elements in $\mathscr{B}(c)$. This contradicts the assumption that $A \in \mathscr{E}(c)$.

Now that C is a convex polytope in $\mathbb{R}^m$, we need at least m equalities among the $m+1$ inequalities

$$x_1 \geqslant \cdots \geqslant x_m \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{m} c_i x_i \leqslant 1$$

to determine an extreme point. Since $\sum_{i=1}^{m} c_i \sigma_i(A) = 1$, there are at most one strict inequality among the inequalities

$$\sigma_1(A) \geqslant \cdots \geqslant \sigma_m(A) \geqslant 0.$$

Consequently, the nonzero singular values of A must all be equal. Hence $A \in (\sum_{i=1}^{j} c_i)^{-1} \mathscr{S}_j$ for some $j = 1, \dots, m$. ∎

For simplicity, we let

$$s_j = \left(\sum_{i=1}^{j} c_i \right)^{-1} \qquad \text{for} \quad j = 1, \dots, m.$$

Let $U \in \mathscr{U}_m$ and $V \in \mathscr{U}_n$. Since $A \in \mathscr{E}(c)$ if and only if $UAV \in \mathscr{E}(c)$, what Lemma 1 says is that $\mathscr{E}(c)$ is made up of finitely many components, each of which is of the form $s_j \mathscr{S}_j$ for some $j = 1, \dots, m$. The following lemma tells us which $s_j \mathscr{S}_j$ is not a component of $\mathscr{S}(c)$.

Lemma 2. *Let $c = (c_1, \dots, c_m) \in \mathbb{R}^m_+ \downarrow$.*

(a) *If $c_1 = \cdots = c_k$ and $2 \leqslant j \leqslant k$, then $(\sum_{i=1}^{j} c_i)^{-1} \mathscr{S}_j \not\subset \mathscr{E}(c)$.*
(b) *If $c_{k+1} = \cdots = c_m = 0$ and $k \leqslant j < m$, then $(\sum_{i=1}^{j} c_i)^{-1} \mathscr{S}_j \not\subset \mathscr{E}(c)$.*

Proof. (a): Suppose $c_1 = \cdots = c_k$ and $2 \leqslant j \leqslant k$. Then $s_j J_j = (A+B)/2$, where

$$A = s_j(J_j + E_{11} - E_{22}) \quad \text{and} \quad B = s_j(J_j - E_{11} + E_{22})$$

are in $\mathscr{B}(c)$. Hence $s_j J_j \notin \mathscr{E}(c)$ and thus $s_j \mathscr{S}_j \not\subset \mathscr{E}(c)$.

(b): Suppose $c_{k+1} = \cdots = c_m = 0$ and $k \leqslant j < m$. Then $s_j J_j = (A+B)/2$, where

$$A = s_j(J_j + E_{mm}) \quad \text{and} \quad B = s_j(J_j - E_{mm})$$

are in $\mathscr{B}(c)$. Hence $s_j J_j \notin \mathscr{E}(c)$ and thus $s_j \mathscr{S}_j \not\subset \mathscr{E}(c)$. ∎

We need Lemmas 3 to 5 to prove Lemma 6, which determines which $s_j\mathscr{S}_j$ is a component of $\mathscr{E}(c)$. Lemma 3 is due to Fan (see [2]).

LEMMA 3. *Let $P=(p_{ij})\in \mathbb{F}_{m\times n}$ have singular values $\sigma_1(P)\geqslant \cdots \geqslant \sigma_m(P)$. Then for any permutation π of the set $\{1,\dots,m\}$ and any $c=(c_1,\dots,c_m)\in \mathbb{R}^m_+\downarrow$, we have*

(a) $\sum_{i=1}^k |p_{\pi(i)\pi(i)}|\leqslant \sum_{i=1}^k \sigma_i(P)$ *for $i=1,\dots,k$, and*
(b) $\sum_{i=1}^k c_i|p_{\pi(i)\pi(i)}|\leqslant \sum_{i=1}^k c_i\sigma_i(P)$ *for $i=1,\dots,k$.*

LEMMA 4. *Let $P=(p_{ij})\in \mathbb{F}_{m\times n}$ be such that*

$$p_{ii}=\sigma_i(P) \qquad \text{for all} \quad i=1,\dots,m,$$

where $\sigma_1(P),\dots,\sigma_m(P)$ are singular values of P. Then

$$P=\sum_{i=1}^m \sigma_i(P)E_{ii}.$$

Proof. Since $\sum_{i=1}^m \sigma_i^2(P)=\operatorname{tr}(PP^*)$, we have

$$\sum_{i=1}^m \sigma_i^2(P)=\sum_{i=1}^m\sum_{j=1}^n |p_{ij}|^2=\sum_{i=1}^m |p_{ii}|^2+\sum_{i\neq j}|p_{ij}|^2$$

$$=\sum_{i=1}^m \sigma_i^2(P)+\sum_{i\neq j}|p_{ij}|^2.$$

This implies $p_{ij}=0$ for all $i\neq j$, and hence the result. ■

LEMMA 5. *Let $k\geqslant 2$, $c_1\geqslant \cdots \geqslant c_k$, and $c_1>c_k$. If*

$$r_1\geqslant \cdots \geqslant r_k, \tag{4}$$

$$\sum_{j=1}^i r_j\geqslant 0 \qquad \text{for} \quad i=1,\dots,k, \tag{5}$$

and

$$\sum_{i=1}^{k} c_i r_i = 0,$$

then

$$r_1 = r_2 = \cdots = r_k = 0.$$

Proof. Note that

$$0 = \sum_{i=1}^{k} c_i r_i = \sum_{i=1}^{k-1} (c_i - c_{i+1}) \sum_{j=1}^{i} r_j + c_k \sum_{j=1}^{k} r_j. \tag{6}$$

Since $c_i - c_{i+1}$, $\sum_{j=1}^{i} r_j$, and c_k are all nonnegative, (6) implies

$$\sum_{j=1}^{i} r_j = 0 \qquad \text{whenever} \quad c_i > c_{i+1}.$$

As $c_1 > c_k$, there must be l such that $1 \leqslant l < k$ and $c_l > c_{l+1}$. Hence

$$\sum_{j=1}^{l} r_j = 0. \tag{7}$$

(7) and (5) give

$$\sum_{j=l+1}^{k} r_j \geqslant 0. \tag{8}$$

We cannot have $r_1 > 0$; otherwise we must have $r_l < 0$ by (4) and (7), and hence $\sum_{j=l+1}^{k} r_j < 0$. This contradicts (8). Thus $r_1 \leqslant 0$ and hence $r_j \leqslant 0$ for all $j = 1, \ldots, k$. By (5), we must have $r_1 = \cdots = r_m = 0$. ∎

LEMMA 6. *Let $c = (c_1, \ldots, c_m) \in \mathbb{R}_+^m \downarrow$ and $1 \leqslant k \leqslant m$ be such that*

$$c_1 > c_k \qquad \textit{if} \quad k \geqslant 2 \tag{9}$$

and

$$c_{k+1} > 0 \qquad \text{if} \quad k < m. \tag{10}$$

Then

$$\left(\sum_{i=1}^{k} c_i \right)^{-1} \mathscr{S}_k \subset \mathscr{E}(c).$$

Proof. Let $s_k J_k = (P+Q)/2$ for some $P = (p_{ij}), Q = (q_{ij}) \in \mathscr{B}(c)$. Then

$$p_{ii} + q_{ii} = \begin{cases} 2s_k & \text{if} \quad i = 1, \ldots, k, \\ 0 & \text{if} \quad i > k. \end{cases} \tag{11}$$

Consider the inequalities

$$\begin{aligned} 2 = \sum_{i=1}^{k} c_i(2s_k) &= \sum_{i=1}^{k} c_i(p_{ii} + q_{ii}) \\ &\leqslant \sum_{i=1}^{k} c_i(|p_{ii}| + |q_{ii}|) && (12) \\ &\leqslant \sum_{i=1}^{k} c_i(\sigma_i(P) + \sigma_i(Q)) && (13) \\ &\leqslant \sum_{i=1}^{m} c_i[\sigma_i(P) + \sigma_i(Q)] && (14) \\ &= \|P\|_c + \|Q\|_c \leqslant 2. \end{aligned}$$

Note that the inequality (13) is by Lemma 3(b). From the above, we see that (12), (13), and (14) must be equalities. Let

$$r_i = \sigma_i(P) + \sigma_i(Q) - 2s_k \qquad \text{for} \quad i = 1, \ldots, k.$$

Then

$$r_1 \geqslant \cdots \geqslant r_k. \tag{15}$$

Also, for $i=1,\dots,k$,

$$\begin{aligned}\sum_{j=1}^{i} r_j &= \sum_{j=1}^{i}\left[\sigma_j(P)+\sigma_j(Q)-2s_k\right]\\ &= \sum_{j=1}^{i}\sigma_j(P)+\sigma_j(Q)-(p_{jj}+q_{jj})\\ &\geqslant \sum_{j=1}^{i}\left[\sigma_j(P)-|p_{jj}|\right]+\left[\sigma_j(Q)-|q_{jj}|\right].\end{aligned}$$

By Lemma 3(a), we then have

$$\sum_{j=1}^{i} r_j \geqslant 0 \qquad \text{for} \quad i=1,\dots,k. \tag{16}$$

Since (12) and (13) are equalities, we also have

$$\begin{aligned}\sum_{i=1}^{k} c_i r_i &= \sum_{i=1}^{k} c_i\left[\sigma_i(P)+\sigma_i(Q)-2s_k\right]\\ &= \left(\sum_{i=1}^{k} c_i\left[\sigma_i(P)+\sigma_i(Q)\right]\right)-2\\ &= 0.\end{aligned} \tag{17}$$

If $k=1$, then (17) gives $r_1=0$ immediately. If $k\geqslant 2$, then (9), (15), (16), (17) and Lemma 5 together give $r_1=\cdots=r_k=0$. Hence, in all cases,

$$\sigma_i(P)+\sigma_i(Q)=2s_k \qquad \text{for} \quad i=1,\dots,k. \tag{18}$$

Since $\sigma_i(P)$ and $\sigma_i(Q)$ $(i=1,\dots,m)$ are in nonincreasing order, by the fact that

$$\sum_{i=1}^{m} c_i\sigma_i(P),\ \sum_{i=1}^{m} c_i\sigma_i(Q)\leqslant 1, \tag{19}$$

(18) and the assumption (10) will force

$$\sigma_i(P) = \sigma_i(Q) = \begin{cases} s_k & \text{if } i = 1, \dots, k, \\ 0 & \text{if } i > k. \end{cases} \tag{20}$$

Let π be a permutation on $\{1, \dots, k\}$ such that if

$$x_i = |p_{\pi(i)\pi(i)}| + |q_{\pi(i)\pi(i)}| \qquad \text{for} \quad i = 1, \dots, k,$$

then $x_i \leqslant \cdots \leqslant x_k$. Now let

$$\begin{aligned} r_i &= \sigma_i(P) + \sigma_i(Q) - x_i \\ &= 2s_k - x_i \qquad \text{for} \quad i = 1, \dots, k. \end{aligned}$$

Then

$$r_1 \geqslant \cdots \geqslant r_k. \tag{21}$$

By Lemma 3(a), we have also

$$\begin{aligned} \sum_{j=1}^{i} r_j &= \sum_{j=1}^{i} \left[\sigma_j(P) - |p_{\pi(j)\pi(j)}|\right] + \left[\sigma_j(Q) - |q_{\pi(j)\pi(j)}|\right] \\ &\geqslant 0 \qquad \text{for} \quad i = 1, \dots, k. \end{aligned} \tag{22}$$

Since (13) is an equality, we get

$$\begin{aligned} \sum_{i=1}^{k} c_i r_i &= \sum_{i=1}^{k} c_i \left[\sigma_i(P) + \sigma_i(Q) - x_i\right] \\ &= \sum_{i=1}^{k} c_i \left[\sigma_i(P) + \sigma_i(Q)\right] - \sum_{i=1}^{k} c_i(|p_{ii}| + |q_{ii}|) \\ &= 0. \end{aligned} \tag{23}$$

If $k = 1$ then (23) gives $r_1 = 0$. If $k \geqslant 2$ then (9), (21), (22), (23), and Lemma

5 together give $r_1 = \cdots = r_k = 0$. Hence

$$|p_{ii}| + |q_{ii}| = 2s_k \quad \text{for} \quad i = 1, \dots, k. \tag{24}$$

By (20), (24) and Lemma 3(a), we get

$$|p_{ii}| = |q_{ii}| = \sigma_i(P) = \begin{cases} s_k & \text{if} \quad i = 1, \dots, k, \\ 0 & \text{if} \quad i > k. \end{cases} \tag{25}$$

Finally, (11) and (24) imply

$$p_{ii} = |p_{ii}| = \sigma_i(P) = \begin{cases} s_k & \text{if} \quad i = 1, \dots, k, \\ 0 & \text{if} \quad i > k. \end{cases}$$

Hence, by Lemma 4,

$$P = \sum_{i=1}^{m} \sigma_i(P) E_{ii} = s_k J_k.$$

Therefore $s_k J_k \in \mathscr{E}(c)$ and hence $s_k \mathscr{S}_k \subset \mathscr{E}(c)$. ∎

Combining Lemmas 1, 2, and 6, we determine the structure of $\mathscr{E}(c)$ in the following theorem.

THEOREM 2. *Let $c = (c_1, \dots, c_m) \in \mathbf{R}^m_+ \downarrow$ be such that exactly k of its first entries are equal and exactly l of its last entries are zero ($1 \leqslant k \leqslant m$, $0 \leqslant l \leqslant m-1$). Denote $s_j = (\sum_{i=1}^{j} c_i)^{-1}$ for $j = 1, \dots, m$. Then the table below describes the structure of $\mathscr{E}(c)$:*

	If k, l satisfies	*then $\mathscr{E}(c) =$*
(a)	$l = m-1$	$s_m \mathscr{S}_m$
(b)	$k = m$	$s_1 \mathscr{S}_1$
(c)	$k < m$, $l < m-1$, $k + l \geqslant m-1$	$s_1 \mathscr{S}_1 \cup s_m \mathscr{S}_m$
(d)	$k + l \leqslant m-2$	$s_1 \mathscr{S}_1 \cup s_m \mathscr{S}_m \cup \left(\bigcup_{j=k+1}^{n-l-1} s_j \mathscr{S}_j \right)$.

In particular,

$$\mathscr{E}_1 = m^{-1}\mathscr{S}_m,$$

$$\mathscr{E}_m = \mathscr{S}_1,$$

and

$$\mathscr{E}_k = \mathscr{S}_1 \cup m^{-1}\mathscr{S}_m \qquad \text{if} \quad 1 < k < m.$$

When $\mathbb{F} = \mathbb{C}$, since each of $\mathscr{U}_m$ and $\mathscr{U}_n$ is path connected, we can easily see that each $\mathscr{S}_j$ $(1 \leqslant j \leqslant m)$ is a connected set. When $\mathbb{F} = \mathbb{R}$, then $\mathscr{U}_k$ has two connected cmponents. As a result, each $\mathscr{S}_j$ $(1 \leqslant j \leqslant m)$ is a connected set except for the case $j = m = n$, in which $\mathscr{S}_m$ $(= \mathscr{U}_m)$ is a union of two connected components

$$\mathscr{S}_m^+ = \left\{ A \in \mathbb{R}_{m\times m} : AA^t = I, \det A = 1 \right\}$$

and

$$\mathscr{S}_m^- = \left\{ A \in \mathbb{R}_{m\times m} : AA^t = I, \det A = -1 \right\}.$$

In any case, $\mathscr{E}(c)$ is a union of finitely many connected components. If T is an isometry from $(\mathbb{F}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{F}_{m\times n}, \|\cdot\|_d)$, then $T(\mathscr{B}(c)) = \mathscr{B}(d)$ and hence $T(\mathscr{E}(c)) = \mathscr{E}(d)$. Since T is one-one and continuous, T acts as a bijection between the set of connected components of $\mathscr{E}(c)$ and that of $\mathscr{E}(d)$. These observations will help us in proving the reuslts in later sections.

4. THE COMPLEX CASE

In this section we let $\mathbb{F} = \mathbb{C}$. Our main results are the following two theorems.

THEOREM 3. *Let $c, d \in \mathbb{R}_+^m\downarrow$. Then the following conditions are equivalent:*

(a) *c and d are scalar multiples of each other.*
(b) *There exists an isometry from $(\mathbb{C}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{C}_{m\times n}, \|\cdot\|_d)$.*
(c) *There exists a linear operator $T : \mathbb{C}_{m\times n} \to \mathbb{C}_{m\times n}$ which maps $\mathscr{S}(d)$ onto $\mathscr{S}(c)$.*

THEOREM 4. *Let $c=(c_1,\dots,c_m)$, $d=(d_1,\dots,d_m)\in\mathbf{R}^m_+\downarrow$ be scalar multiples of each other, and $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ be linear. Then the following conditions are equivalent:*

(a) *T is an isometry from $(\mathbf{C}_{m\times n},\|\cdot\|_c)$ to $(\mathbf{C}_{m\times n},\|\cdot\|_d)$.*

(b) *$T(\mathscr{S}(d))=\mathscr{S}(c)$.*

(c) *There exist $U=\mathscr{U}_m$ and $V\in\mathscr{U}_n$ such that either*

  (i) *$T(A)=(c_1/d_1)UAV$ for all A, or*

  (ii) *$m=n$ and $T(A)=(c_1/d_1)UA^tV$ for all A.*

We divide the proofs of these theorems into several lemmas.

LEMMA 7. *Let $1<k<m$ and $r>0$. Then there is no linear operator $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ such that $T(\mathscr{S}_1)=r\mathscr{S}_k$.*

Proof. Without loss of generality we may let $r=1$; otherwise replace T by $r^{-1}T$. Suppose there exists $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ such that $T(\mathscr{S}_1)=\mathscr{S}_k$. By Theorem 1, T^* is an isometry from $(\mathbf{C}_{m\times n},\|\cdot\|_k)$ to $(\mathbf{C}_{m\times n},\|\cdot\|_1)$. It then follows that $T^*(\mathscr{B}_k)=\mathscr{B}_1$ and hence $T^*(\mathscr{E}_k)=\mathscr{E}_1$. However, by Theorem 2, $\mathscr{E}_k$ and $\mathscr{E}_1$ have different numbers of connected components. Hence T cannot exist. ∎

LEMMA 8. *Let $r>0$. Then there is no linear operator $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ such that $T(\mathscr{S}_1)\subset r\mathscr{S}_m$.*

Proof. Suppose there exists such a T. We may assume $r=1$ without loss of generality; otherwise replace T by $r^{-1}T$. Notice that $A\in\mathscr{S}_m$ if and only if A has orthonormal rows, i.e., $AA^*=I_m$. Let $T(E_{1j})=A_j$ for each $j=1,\dots,n$. Then

$$A_jA_j^*=I_m \qquad \text{for} \quad j=1,\dots,n,$$

as $T(\mathscr{S}_1)=\mathscr{S}_m$. For any $1\leqslant i,j\leqslant n$ with $i\neq j$, and any $a,b\in\mathbf{C}$ with $|a|^2+|b|^2=1$, since $aE_{1i}+bE_{1j}\in\mathscr{S}_1$, we have $aA_i+bA_j=T(aE_{1i}+bE_{1j})\in\mathscr{S}_m$. Therefore

$$\begin{aligned} I_m&=(aA_i+bA_j)(aA_i+bA_j)^*\\ &=|a|^2A_iA_i^*+|b|^2A_jA_j^*+a\bar{b}A_iA_j^*+\bar{a}bA_jA_i^*, \end{aligned}$$

which implies that

$$0 = a\bar{b}A_iA_j^* + \bar{a}bA_jA_i^*.$$

By taking $a = b = 2^{-1/2}$ and $a = b\sqrt{-1} = 2^{-1/2}$ in turn, we get

$$A_iA_j^* = 0.$$

This means each row of A_i is orthogonal to each row of A_j. As a result, the collection of all the (mn) rows of the matrices $A_1, \ldots, A_n$ form an orthonormal set of vectors in $\mathbf{C}^n$. This is absurd, since $m \geqslant 2$. Hence T cannot exist. ∎

LEMMA 9. *Let $r > 0$ and $T: \mathbf{C}_{m\times n} \to \mathbf{C}_{m\times n}$ be linear. Then T satisfies one of the conditions*

(i) *there exist $U \in \mathscr{U}_m$ and $V \in \mathscr{U}_n$ such that*

$$T(A) = rUAV \qquad \text{for all } A, \quad \text{or}$$

(ii) *$m = n$ and there exist $U, V \in \mathscr{U}_m$ such that*

$$T(A) = rUA^tV \qquad \text{for all } A,$$

if and only if its dual transformation T^* satisfies the same condition.

Proof. Notice that

$$T(A) = rUAV \qquad \text{for all } A$$

if and only if

$$T^*(A) = rU^*AV^* \qquad \text{for all } A,$$

and if $m = n$,

$$T(A) = rUA^tV \qquad \text{for all } A,$$

if and only if

$$T^*(A) = r\bar{V}A^t\bar{U} \qquad \text{for all } A,$$

where $\bar{U}$ and $\bar{V}$ are the complex conjugates of U and V respectively. Hence the result follows. ∎

LEMMA 10. *Let $c=(c_1,\dots,c_m)$, $d=(d_1,\dots,d_m)\in\mathbf{R}^m_+\downarrow$. Suppose T is an isometry from $(\mathbf{C}_{m\times n},\|\cdot\|_c)$ to $(\mathbf{C}_{m\times n},\|\cdot\|_d)$. Then both T and T^* satisfy condition* (c) *of Theorem* 4.

Proof. Let T be an isometry from $(\mathbf{C}_{m\times n},\|\cdot\|_c)$ to $(\mathbf{C}_{m\times n},\|\cdot\|_d)$. Then T acts as a bijection between the set of connected components of $\mathscr{E}(c)$ and that of $\mathscr{E}(d)$. Suppose $c_1^{-1}\mathscr{S}_1$ is a component of $\mathscr{E}(c)$. Then $T(c_1^{-1}\mathscr{S}_1)$ is a component of $\mathscr{E}(d)$. By Theorem 2 and Lemmas 7 and 8, we see that $d_1^{-1}\mathscr{S}_1$ must be a component of $\mathscr{E}(d)$ and is the image of $c_1^{-1}\mathscr{S}_1$ under T. Therefore $(d_1/c_1)T(\mathscr{S}_1)=\mathscr{S}_1$. By Theorem 1, $(d_1/c_1)T^*$ is an isometry on $(\mathbf{C}_{m\times n},\|\cdot\|_1)$. If $c_1^{-1}\mathscr{S}_1$ is not a component of $\mathscr{E}(c)$, then by Theorem 2, $\mathscr{E}(c)=s_m\mathscr{S}_m$, where $s_m=(\Sigma_{i=1}^m c_i)^{-1}$. By considering T^{-1} instead of T and using Theorem 2 and Lemma 8, we see that $\mathscr{E}(d)$ can have only one component and must be $r_m\mathscr{S}_m$, where $r_m=(\Sigma_{i=1}^m d_i)^{-1}$. Hence $T(s_m\mathscr{S}_m)=r_m\mathscr{S}_m$, which implies $(s_m/r_m)T(\mathscr{S}_m)=\mathscr{S}_m$. By Theorem 1, $(s_m/r_m)T^*$ is an isometry on $(\mathbf{C}_{m\times n},\|\cdot\|_m)$. Hence, in all cases, there is $r>0$ such that $r^{-1}T^*$ is an isometry on either $(\mathbf{C}_{m\times n},\|\cdot\|_1)$ or $(\mathbf{C}_{m\times n},\|\cdot\|_m)$. By the known result on the characterization of isometries on $(\mathbf{C}_{m\times n},\|\cdot\|_1)$ or on $(\mathbf{C}_{m\times n},\|\cdot\|_m)$ (see [6, 5]), there exist $U\in\mathscr{U}_m$ and $V\in\mathscr{U}_n$ such that either

$$r^{-1}T^*(A)=UAV\qquad\text{for all } A,$$

or

$$m=n\quad\text{and}\quad r^{-1}T^*(A)=UA^tV\quad\text{for all } A.$$

Hence T^* satisfies condition (i) or (ii) of Lemma 9. By Lemma 9, T also satisfies condition (i) or (ii). By taking $A=E_{11}$, we get

$$c_1=\|E_{11}\|_c=\|T(E_{11})\|_d=\|rUE_{11}V\|_d=rd_1.$$

Therefore $r=c_1/d_1$. Hence T and T^* satisfy condition (c) of Theorem 4. ∎

We are now ready to give the

Proof of Theorem 3. (b) $\Leftrightarrow$ (c) is by Theorem 1.

(a) $\Rightarrow$ (b): Suppose $c = rd$, where $r > 0$. Then $T: \mathbf{C}_{m\times n} \to \mathbf{C}_{m\times n}$ defined by

$$T(A) = rA \qquad \text{for all } A$$

is an isometry from $(\mathbf{C}_{m\times n}, \|\cdot\|_c)$ to $(\mathbf{C}_{m\times n}, \|\cdot\|_d)$.

(b) $\Rightarrow$ (a): Suppose T is an isometry from $(\mathbf{C}_{m\times n}, \|\cdot\|_c)$ to $(\mathbf{C}_{m\times n}, \|\cdot\|_d)$. By Lemma 10, T satisfies condition (c) of Theorem 4. By Lemma 9, T^* satisfies the same condition. Since $T^*(\mathscr{S}(d)) = \mathscr{S}(c)$ by Theorem 1, we have

$$\mathscr{S}(c) = T^*(\mathscr{S}(d)) = \frac{c_1}{d_1} U(\mathscr{S}(d))V = \frac{c_1}{d_1}\mathscr{S}(d).$$

It follows that $c = (c_1/d_1)d$. ∎

Proof of Theorem 4. (a) $\Rightarrow$ (c) is by Lemma 10. (c) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are by direct verification.

(b) $\Rightarrow$ (c): Suppose T satisfies (b). Then T^* is an isometry from $(\mathbf{C}_{m\times n}, \|\cdot\|_c)$ to $(\mathbf{C}_{m\times n}, \|\cdot\|_d)$ by Theorem 1. By Lemma 10, $T = (T^*)^*$ satisfies condition (c). ∎

5. THE REAL CASE

In this section we let $\mathbf{F} = \mathbf{R}$. Before stating the results, we present two particular linear operators $L, \mathscr{L}: \mathbf{R}_{4\times 4} \to \mathbf{R}_{4\times 4}$ which are defined by

$$L(A) = \tfrac{1}{2}(A + B_1AC_1 + B_2AC_2 + B_3AC_3) \qquad \text{for all } A$$

and

$$\mathscr{L}(A) = -D[L(DA)] \qquad \text{for all } A,$$

where

$$B_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad B_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

$$B_3=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$C_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad C_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$C_3=\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$

$$D=\operatorname{diag}(1,-1,-1,-1).$$

The interesting properties of L and $\mathscr{L}$ are discussed in [8]. Here we state the following facts about L and $\mathscr{L}$ that are useful in our discussion:

(I) Both L and $\mathscr{L}$ are self-adjoint, i.e., $L^*=L$ and $\mathscr{L}^*=\mathscr{L}$.

(II) If $A\in\mathbf{R}_{4\times4}$ has singular values $a_1\geq a_2\geq a_3\geq a_4$, then both $L(A)$ and $\mathscr{L}(A)$ have singular values $(d_1+d_2+d_3+\delta d_4)/2$, $(d_1+d_2-d_3-\delta d_4)/2$, $(d_1-d_2+d_3-\delta d_4)/2$, and $|(d_1-d_2-d_3+\delta d_4)|/2$, where

$$\delta=\begin{cases}1 & \text{if } \det(A)\geq 0,\\ -1 & \text{otherwise.}\end{cases}$$

(III) $T:\mathbf{R}_{4\times4}\to\mathbf{R}_{4\times4}$ is an isometry on $(\mathbf{R}_{4\times4},\|\cdot\|_2)$ and satisfies $T(\mathscr{S}_1)\neq\mathscr{S}_1$ if and only if there exist $U,V\in\mathscr{U}_4$ such that one of the following holds:

(i) $T(A)=L(UAV)$ for all A;
(ii) $T(A)=L(UA^tV)$ for all A;
(iii) $T(A)=\mathscr{L}(UAV)$ for all A;
(iv) $T(A)=\mathscr{L}(UA^tV)$ for all A.

As counterparts of Theorems 3 and 4, we have the following results for the real case.

THEOREM 5. *Let $c, d \in \mathbb{R}^m_+\downarrow$. Then the following conditions are equivalent:*

(a) *c and d are scalar multiples of each other.*
(b) *There exists an isometry from $(\mathbb{R}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{R}_{m\times n}, \|\cdot\|_d)$.*
(c) *There exists a linear operator $T:\mathbb{R}_{m\times n}\to\mathbb{R}_{m\times n}$ which maps $\mathscr{S}(d)$ onto $\mathscr{S}(c)$.*

THEOREM 6. *Let $c=(c_1,\dots,c_m)$, $d=(d_1,\dots,d_m)\in\mathbb{R}^m_+\downarrow$ be scalar multiples of each other, and $T:\mathbb{R}_{m\times n}\to\mathbb{R}_{m\times n}$ be linear. Then the following conditions are equivalent:*

(a) *T is an isometry from $(\mathbb{R}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{R}_{m\times n}, \|\cdot\|_d)$.*
(b) *$T(\mathscr{S}(d)) = \mathscr{S}(c)$.*
(c) *There exist $U\in\mathscr{U}_m$ and $V\in\mathscr{U}_n$ such that one of the following holds:*
   (i) *$T(A) = (c_1/d_1)UAV$ for all A;*
   (ii) *$m=n$ and $T(A)=(c_1/d_1)UA^tV$ for all A;*
   (iii) *$m=n=4$, $c_1=c_2+c_3$, $c_4=0$, and*

$$T(A) = T_1\circ T_2(A) \qquad \text{for all } A,$$

where T_1 is L or $\mathscr{L}$, and T_2 is in one of the forms described in (i) *or* (ii).

Condition (c) of Theorem 6 is more complicated than that of Theorem 4, and the proof is more involved. This is due to the fact that isometries on $\mathbb{R}_{4\times 4}$ may have more complicated structure. Again we divide the proofs into a number of lemmas.

LEMMA 11. *Suppose $1<k<m$ and $r>0$. Then there is no linear operator $T:\mathbb{R}_{m\times n}\to\mathbb{R}_{m\times n}$ such that $T(\mathscr{S}_1)=r\mathscr{S}_k$.*

Proof. Same as that of Lemma 7. ∎

LEMMA 12. *Let $(m,n)\neq(4,4)$ and $r>0$. Then there is no nonsingualr linear operator $T:\mathbb{R}_{m\times n}\to\mathbb{R}_{m\times n}$ such that $T(\mathscr{S}_1)=r\mathscr{S}'_m$, where $\mathscr{S}'_m$ is a connected component of $\mathscr{S}_m$.*

Proof. Suppose there exist such a T. We may assume $r=1$ without loss of generality. Let $X=(x_{ij})\in\mathbb{R}_{m\times n}$ be such that

$$f(X,h)=E_{11}+hX+\alpha(X,h)\in\mathcal{S}_1, \tag{27}$$

where h is any small real number and $\alpha(X,h)\in\mathbb{R}_{m\times n}$ has entries $o(h^2)$. For any i,j such that $2\leqslant i\leqslant m$ and $2\leqslant j\leqslant n$, let A be the 2×2 matrix formed by the $(1,1),(1,j),(i,1),(i,j)$ entries of $f(X,h)$, i.e.,

$$A=\begin{pmatrix} 1+hx_{11}+o(h^2) & hx_{1j}+o(h^2)\\ hx_{i1}+o(h^2) & hx_{ij}+o(h^2)\end{pmatrix}.$$

Then A has rank 1 and $\operatorname{tr}AA^t\leqslant 1$. Hence $x_{11}=x_{ij}=0$. Let

$$\mathcal{V}=\operatorname{span}\{E_{ij}: i=1 \text{ or } j=1, (i,j)\neq(1,1)\}.$$

Then $X\in\mathcal{V}$. On the other hand, if $X\in\mathcal{V}$, then we can find $\alpha(X,h)\in\mathbb{R}_{m\times n}$ of $o(h^2)$ such that (27) is satisfied for all small h. Hence $\mathcal{V}$ is the tangent space to $\mathcal{S}_1$ at E_{11}.

Now consider $Y=(y_{ij})\in\mathbb{R}_{m\times n}$ such that

$$g(Y,h)=\sum_{i=1}^{m}E_{ii}+hY+\beta(Y,h)\in\mathcal{S}_m, \tag{28}$$

where h is any small real number and $\beta(Y,h)\in\mathbb{R}_{m\times n}$ is $o(h^2)$. Since $g(Y,h)g(Y,h)^t=I_m$, we have

$$\left(\sum_{i=1}^{m}E_{ii}\right)Y^t+Y\left(\sum_{i=1}^{m}E_{ii}\right)^t=0.$$

As a result,

$$Y=[Y_1|Y_2]$$

where $Y_1=-Y_1^t\in\mathbb{R}_{m\times m}$ and $Y_2\in\mathbb{R}_{m\times(n-m)}$. Conversely, any matrix Y in this form will yield a function $\beta(Y,h)$ of $o(h^2)$ which satisfies (28). Thus

$$\mathcal{W}=\left\{[Y_1|Y_2]\in\mathbb{R}_{m\times n}: Y_1=-Y_1^t\in\mathbb{R}_{m\times m}, Y_2\in\mathbb{R}_{m\times(n-m)}\right\}$$

is the tangent space of $\mathscr{S}_m$ at $\sum_{i=1}^m E_{ii}$. As $T(\mathscr{S}_1)=\mathscr{S}_m'$, we may assume $T(E_{11})=\sum_{i=1}^m E_{ii}$; otherwise replace T by T' where $T'(A)=UT(A)V$ for all A, and U,V are suitable fixed elements in $\mathscr{U}_m$ and $\mathscr{U}_n$ respectively [if $m=n$ and $T(\mathscr{S}_1)=\mathscr{S}_m^-$, the chosen $U,V\in\mathscr{U}_m$ must satisfy $\det(UV)=-1$]. As a result, T must map the tangent space $\mathscr{V}$ onto $\mathscr{W}$. Then $\dim\mathscr{V}=\dim\mathscr{W}$ because T is nonsingular. Notice that $\dim\mathscr{V}=n+m-2$ and $\dim\mathscr{W}=m(m-1)/2+m(n-m)$. If $n=m\neq 4$, then

$$\dim\mathscr{W}=\frac{m(m-1)}{2}\neq 2(m-1)=\dim\mathscr{V}.$$

If $n>m$, let $n=m+k$. Then

$$\dim\mathscr{W}-\dim\mathscr{V}=(m-1)\left(\frac{m}{2}+k-2\right),$$

which is zero if and only if $m=2$ and $k=1$. Hence T can exist only if $(m,n)=(2,3)$. We now show that T cannot exist when $(m,n)=(2,3)$. Notice that $A\in\mathscr{S}_2$ if and only if A has orthonormal rows, and we have assumed that $T(E_{11})=E_{11}+E_{22}$. Let $T(E_{12})=A(\in\mathscr{S}_2)$ and $s=2^{-1/2}$. Since $s(E_{11}+E_{12})\in\mathscr{S}_1$, $s(E_{11}+E_{22}+A)=T(s(E_{11}+E_{12}))\in\mathscr{S}_2$. Hence

$$I_2=\tfrac{1}{2}(E_{11}+E_{22}+A)(E_{11}+E_{22}+A)^t$$

$$\Rightarrow\quad (E_{11}+E_{22})A^t+A(E_{11}+E_{22})^t=0$$

$$\Rightarrow\quad A=\begin{pmatrix}0 & x & y\\ -x & 0 & z\end{pmatrix}$$

for some real numbers x, y, and z. Since A has orthonormal rows, we get $x=\pm 1$, $y=z=0$. Similarly, we can prove that

$$T(E_{13})=\pm\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}.$$

Then $T(E_{13})=\pm T(E_{12})$. Hence T cannot be one-one. So T cannot exist. ∎

We remark that in the proof of Lemma 12, one may count the dimension of the algebraic sets $\mathscr{S}_1$ and $\mathscr{S}_m$ instead of considering the tangent spaces in order to get the conclusion. Nevertheless, the proof we present is elementary.

LEMMA 13. *Let $T:\mathbf{R}_{m\times n}\to\mathbf{R}_{m\times n}$ be linear. Then T satisfies $T(\mathcal{S}_1)=\mathcal{S}_1$ if and only if there exist $U\in\mathcal{U}_m$ and $V\in\mathcal{U}_n$ such that either*

(i) *$T(A)=UAV$ for all A, or*
(ii) *$m=n$ and $T(A)=UA^tV$ for all A.*

Proof. If T satisfies condition (i) or (ii), then $T(\mathcal{S}_1)=\mathcal{S}_1$ by direct verification. Conversely, suppose $T(\mathcal{S}_1)=\mathcal{S}_1$. Then $T(\mathcal{R}_1)=\mathcal{R}_1$, where $\mathcal{R}_1$ is the set of all rank-1 matrices in $\mathbf{R}_{m\times n}$. By a result in [13] (which deals with a more general problem; see also [1]), there are nonsingular $M\in\mathbf{R}_{m\times m}$ and $N\in\mathbf{R}_{n\times n}$ such that either

$$T(A)=MAN \qquad \text{for all } A \tag{29}$$

or

$$m=n \quad \text{and} \quad T(A)=MA^tN \quad \text{for all } A. \tag{30}$$

Let M and N have singular value decomposition

$$M=U_1\operatorname{diag}(a_1,\dots,a_m)U_2, \qquad N=V_1\operatorname{diag}(b_1,\dots,b_n)V_2,$$

where $U_1,U_2\in\mathcal{U}_m$, $V_1,V_2\in\mathcal{U}_n$, $a_1\geqslant\cdots\geqslant a_m>0$, and $b_1\geqslant\cdots\geqslant b_n>0$. If T satisfies (29), then for any $1\leqslant i\leqslant m$, $1\leqslant j\leqslant n$, and $B=U_2^{-1}E_{ij}V_1^{-1}\in\mathcal{S}_1$, we have

$$a_ib_jU_1E_{ij}V_2=MBN=T(B)\in\mathcal{S}_1.$$

Thus we may assume $a_1=\cdots=a_m=1=b_1=\cdots=b_n$. Hence $M\in\mathcal{U}_m$, $N\in\mathcal{U}_n$, and T satisfies (i). If T satisfies (30), then replace the above B by its transpose. Using a similar argument, we can show that T satisfies (ii). ∎

LEMMA 14. *Let $r>0$ and $T:\mathbf{R}_{m\times n}\to\mathbf{R}_{m\times n}$ be linear. Then T satisfies one of the conditions*

(i) *there exist $U\in\mathcal{U}_m$, $V\in\mathcal{U}_n$ such that*

$$T(A)=rUAV \qquad \text{for all } A,$$

(ii) *$m = n$ and there exist $U, V \in \mathscr{U}_m$ such that*

$$T(A) = rUA^tV \qquad \text{for all } A$$

or

(iii) *$m = n = 4$ and $T = T_1 \circ T_2$, where $T_1 = L$ or $\mathscr{L}$, and T_2 is in one of the forms described in* (i) *or* (ii)

if and only if its dual transformation T^* satisfies the same condition.

Proof. If T (or T^*, respectively) satisfies condition (i) or (ii), then by the same argument given in the proof of Lemma 9, T^* (or T, respectively) satisfies the same condition. Suppose T satisfies condition (iii), i.e., $T = T_1 \circ T_2$, where $T_1 = L$ or $\mathscr{L}$ and T_2 is in one of the forms described in (i) or (ii). By properties (I) and (III) of L and $\mathscr{L}$, $T_1^* = T_1$ is an isometry on $(\mathbf{R}_{4\times 4}, \|\cdot\|_2)$ and $T_1(\mathscr{S}_1) \neq \mathscr{S}_1$. Since T_2 (and hence T_2^*) satisfies condition (i) or (ii) of the lemma, $r^{-1}T_2^*$ is an isometry on $(\mathbf{R}_{m\times n}, \|\cdot\|_c)$ and maps $\mathscr{S}(c)$ onto $\mathscr{S}(c)$ for all $c \in \mathbf{R}_+^m\downarrow$. As a result, $r^{-1}T^* = r^{-1}T_2^* \circ T_1$ is an isometry on $(\mathbf{R}_{4\times 4}, \|\cdot\|_2)$ and $r^{-1}T^*(\mathscr{S}_1) \neq \mathscr{S}_1$. By fact (III) again, T^* must satisfy condition (iii) of the lemma. The converse can be proved by interchanging T and T^*. ∎

LEMMA 15. *Let $m = n = 4$ and $c = (c_1, \dots, c_4), d = (d_1, \dots, d_4) \in \mathbf{R}_+^4\downarrow$. Suppose T is an isometry from $(\mathbf{R}_{4\times 4}, \|\cdot\|_c)$ to $(\mathbf{R}_{4\times 4}, \|\cdot\|_d)$, $T(\mathscr{S}_1) \neq r\mathscr{S}_1$ for all $r > 0$, and $c_1^{-1}\mathscr{S}_1 \subset \mathscr{E}(c)$. Then $c = (c_1/d_1)d$, and both T and T^* satisfy condition* (c)(iii) *of Theorem* 6.

Proof. Suppose $c_1^{-1}\mathscr{S}_1 \subset \mathscr{E}(c)$, and T is an isometry from $(\mathbf{R}_{4\times 4}, \|\cdot\|_c)$ to $(\mathbf{R}_{4\times 4}, \|\cdot\|_d)$ such that $T(\mathscr{S}_1) \neq s\mathscr{S}_1$ for all $s > 0$. Since T acts as a bijection between the set of connected components of $\mathscr{E}(c)$ and that of $\mathscr{E}(d)$, by Theorem 2 and Lemma 11 we see that $r_4\mathscr{S}_4'$ must be a connected component of $\mathscr{E}(d)$ and is the image of $c_1^{-1}\mathscr{S}_1$ under T, where $r_4 = (\sum_{i=1}^4 d_i)^{-1}$ and $\mathscr{S}_4'$ is a connected component of $\mathscr{S}_4$. Now that $\mathscr{E}(c)$ and $\mathscr{E}(d)$ have the same number of connected components, using similar argument, we also have

$$T(s_4\mathscr{S}_4'') = d_1^{-1}\mathscr{S}_1 \subset \mathscr{E}(d),$$

where $s_4 = (\sum_{i=1}^4 c_i)^{-1}$ and $\mathscr{S}_4''$ is a connected component of $\mathscr{S}_4$. We may assume $\mathscr{S}_4' = \mathscr{S}_4^+$ without loss of generality; otherwise replace T by T', where $T'(A) = \operatorname{diag}(1,1,1,-1)T(A)$ for all A. Similarly we may assume $\mathscr{S}_4'' = \mathscr{S}_4^+$; otherwise replace T by T'' where $T''(A) = T(\operatorname{diag}(1,1,1,-1)A)$

for all A. Now consider $A = U(E_{11}+E_{22})V \in \mathscr{S}_2$ for any $U, V \in \mathscr{U}_4$ with $\det(UV)=1$. Since $A_1 = U\mathrm{diag}(1,1,1,1)V$ and $A_2 = U\mathrm{diag}(1,1,-1,-1)V$ are in $\mathscr{S}_4^+$, $T(A_1)$ and $T(A_2)$ are in $(d_1s_4)^{-1}\mathscr{S}_1$ and hence have rank 1. Then $T(A)=[T(A_1)+T(A_2)]/2$ has rank not exceeding 2. Similarly, if $B = U(E_{12}+E_{21})V$, then $T(B)$ has rank not exceeding 2. Observe that $A \pm B \in 2\mathscr{S}_1$. Therefore

$$T(A) \pm T(B) = T(A \pm B) \in 2c_1r_4\mathscr{S}_4,$$

and hence

$$\begin{aligned} &T(A)T(A)^t + T(B)T(B)^t \pm \left[T(A)T(B)^t + T(B)T(A)^t\right] \\ &\quad = [T(A) \pm T(B)][T(A) \pm T(B)]^t \\ &\quad = 4(c_1r_4)^2 I. \end{aligned}$$

If follows that

$$T(A)T(A)^t + T(B)T(B)^t = 4(c_1r_4)^2 I. \tag{31}$$

Since $T(A)$ has rank at most 2, there is $W \in \mathscr{U}_4$ such that

$$W\left(T(A)T(A)^t\right)W^t = \mathrm{diag}(a,b,0,0),$$

where $a \geqslant b \geqslant 0$. Then (31) implies

$$W\left[T(B)T(B)^t\right]W^t = 4(c_1r_4)^2 I - \mathrm{diag}(a,b,0,0).$$

As $T(B)$ has rank not exceeding 2, we must have $a = b = 4(c_1r_4)^2$. Thus $T(A) \in 2c_1r_4\mathscr{S}_2$. As a result, $T(\mathscr{S}_2) \subset 2c_1r_4\mathscr{S}_2$. By considering T^{-1} instead of T, we get $T^{-1}(\mathscr{S}_2) \subset 2d_1s_4\mathscr{S}_2$. Combining, we get

$$T(\mathscr{S}_2) = r\mathscr{S}_2,$$

where

$$r = 2c_1r_4 = (2d_1s_4)^{-1}.$$

By Theorem 1, $r^{-1}T^*$ is an isometry on $(\mathbb{R}_{4\times 4}, \|\cdot\|_2)$. As $T(\mathscr{S}_1) \neq s\mathscr{S}_1$ for all $s>0$, $T^*(\mathscr{S}_1) \neq s\mathscr{S}_1$ for all $s>0$ also; otherwise $s^{-1}T^*$ will satisfy condition (i) or (ii) of Lemma 13 and, by Lemma 14, $s^{-1}T$ also satisfies condition (i) or (ii) of Lemma 13, and hence $T(\mathscr{S}_1) = s\mathscr{S}_1$, which is a contradiction. By fact (III) on the properties of L and $\mathscr{L}$, and by Lemma 14, both T and T^* satisfy condition (iii) of Lemma 14 for some $r>0$. If $A \in \mathscr{S}(d)$, then by fact (II) on the properties of L and $\mathscr{L}$, $T^*(A)$ will have singular values

$$\frac{r(d_1+d_2+d_3+\delta d_4)}{2}, \qquad \frac{r(d_1+d_2-d_3-\delta d_4)}{2},$$
$$\frac{r(d_1-d_2+d_3-\delta d_4)}{2}, \qquad \frac{|r(d_1-d_2-d_3+\delta d_4)|}{2}, \tag{32}$$

where $\delta = 1$ or -1 according as $\det A \geq 0$ or $\det A < 0$. Since $T^*(\mathscr{S}(d)) = \mathscr{S}(c)$ by Theorem 1, the numbers in (32) must be c_1, c_2, c_3, and c_4 (not necessarily in that order), no matter whether $\delta = 1$ or -1. By direct computations, we get

$$\begin{aligned} d_4 &= 0, \\ c_1 + c_2 &= r(d_1+d_2), \\ c_2 + c_3 &= rd_1, \\ c_4 &= \frac{|r(d_1-d_2-d_3)|}{2}. \end{aligned}$$

Considering T^{-1} instead of T, we get

$$\begin{aligned} c_4 &= 0, \\ d_1 + d_2 &= r^{-1}(c_1+c_2), \\ d_2 + d_3 &= r^{-1}c_1, \\ d_4 &= \frac{|r^{-1}(c_1-c_2-c_3)|}{2}. \end{aligned}$$

As a result, we have $r = c_1/d_1$, $c = rd$, $c_1 = c_2 + c_3$, and $c_4 = 0$. Hence both T and T^* satisfy condition (c)(iii) of Theorem 6. ∎

LEMMA 16. *Let* $c = (c_1, \dots, c_m)$, $d = (d_1, \dots, d_m) \in \mathbb{R}^m_{+\downarrow}$. *If* T *is an isometry from* $(\mathbb{R}_{m\times n}, \|\cdot\|_c)$ *to* $(\mathbb{R}_{m\times n}, \|\cdot\|_d)$, *then* $c = (c_1/d_1)d$ *and both* T *and* T^* *satisfy condition* (c) *of Theorem* 6.

Proof. Let T be an isometry from $(\mathbb{R}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{R}_{m\times n}, \|\cdot\|_d)$. Then T is nonsingular and acts as a bijection between the set of connected components of $\mathscr{E}(c)$ and that of $\mathscr{E}(d)$.

Suppose $c_1^{-1}\mathscr{S}_1 \not\subset \mathscr{E}(c)$. Then, by Theorem 2, $\mathscr{E}(c) = s_m\mathscr{S}_m$, where $s_m = (\sum_{i=1}^m c_i)^{-1}$. Since $\mathscr{E}(c)$ and $\mathscr{E}(d)$ have the same number of connected components, by Theorem 2 again, either

$$m < n \quad \text{and} \quad \mathscr{E}(d) = d_1^{-1}\mathscr{S}_1 \tag{33}$$

or $\mathscr{E}(d) = r_m\mathscr{S}_m$ where $r_m = (\sum_{i=1}^m d_i)^{-1}$. If (33) holds then

$$T^{-1}(\mathscr{S}_1) = d_1T^{-1}(d_1^{-1}\mathscr{S}_1) = d_1T^{-1}(\mathscr{E}(d)) = d_1\mathscr{E}(c) = d_1s_m\mathscr{S}_m.$$

However, if $m < n$, then by Lemma 12 there is no nonsingular linear operator which maps $\mathscr{S}_1$ onto $\mathscr{S}_m$. Hence (33) cannot hold. Thus $\mathscr{E}(d) = r_m\mathscr{S}_m$ and

$$T(s_m\mathscr{S}_m) = T(\mathscr{E}(c)) = \mathscr{E}(d) = r_m\mathscr{S}_m.$$

By Theorem 1, since $(s_m/r_m)T(\mathscr{S}_m) = \mathscr{S}_m$, $(s_m/r_m)T^*$ is an isometry on $(\mathbb{R}_{m\times n}, \|\cdot\|_m)$. As $\mathscr{E}_m = \mathscr{S}_1$, we get

$$\frac{s_m}{r_m}T^*(\mathscr{S}_1) = \mathscr{S}_1.$$

By Lemmas 13 and 14, both T and T^* satisfy condition (i) or (ii) of Lemma 14 for some $r > 0$. Since $T^*(\mathscr{S}(d)) = \mathscr{S}(c)$ by Theorem 1, we have

$$\mathscr{S}(c) = T^*(\mathscr{S}(d)) = rU\mathscr{S}(d)V = r\mathscr{S}(d).$$

This gives $c = rd$ and $r = c_1/d_1$. Hence T and T^* satisfy (c)(i) or (c)(ii) of Theorem 6.

Suppose $c_1^{-1}\mathscr{S}_1 \subset \mathscr{E}(c)$. If $(m,n) \neq (4,4)$, then by Lemmas 11 and 12, we see that $d_1^{-1}\mathscr{S}_1$ must be a component of $\mathscr{E}(d)$ and is the image of $c_1^{-1}\mathscr{S}_1$ under T. Hence $T(\mathscr{S}_1) = r\mathscr{S}_1$ for some $r > 0$. Using Lemma 13 and applying similar arguments to the above, we can show that $c = (c_1/d_1)d$ and that both T and T^* satisfy condition (c)(i) or (c)(ii) of Theorem 6. If $m = n = 4$ and $T(\mathscr{S}_1) \neq r\mathscr{S}_1$ for any $r > 0$, then by Lemma 15, $c = (c_1/d_1)d$ and both T and T^* satisfy condition (c)(iii) of Theorem 6. ∎

We now give the

Proof of Theorem 5. (b) $\Leftrightarrow$ (c) is by Theorem 1.

(a) $\Rightarrow$ (b): If $c = rd$ for some $r > 0$, then $T: \mathbb{R}_{m\times n} \to \mathbb{R}_{m\times n}$ defined by

$$T(A) = rA \qquad \text{for all } A$$

is an isometry from $(\mathbb{R}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{R}_{m\times n}, \|\cdot\|_d)$.

(b) $\Rightarrow$ (a) is by Lemma 16. ∎

Proof of Theorem 6. (a) $\Rightarrow$ (c) is by Lemma 16.

(c) $\Rightarrow$ (b) is by direct verification.

(b) $\Rightarrow$ (a): Suppose T satisfies (b). Then by Theorem 1, T^* is an isometry from $(\mathbb{R}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{R}_{m\times n}, \|\cdot\|_d)$. As (a) implies (c), T^* will satisfy condition (c). By direct verification, $T^*(\mathscr{S}(d)) = \mathscr{S}(c)$. Hence T satisfies (a) by Theorem 1. ∎

6. A VARIATION OF THE COMPLEX CASE

Since $\mathbb{C}_{m\times n}$ is sometimes regarded as a real linear space (of dimension $2mn$), it might be interesting to determine the structure of the real isometries from $(\mathbb{C}_{m\times n}, \|\cdot\|_c)$ to $(\mathbb{C}_{m\times n}, \|\cdot\|_d)$ or the real linear operators on $\mathbb{C}_{m\times n}$ which map $\mathscr{S}(d)$ onto $\mathscr{S}(c)$, where $c = (c_1, \dots, c_m)$, $d = (d_1, \dots, d_m) \in \mathbb{R}_+^m\downarrow$, and

$$\mathscr{S}(c) = \{X \in \mathbb{C}_{m\times n} : X \text{ has singular values } c_1, \dots, c_m\}.$$

Using the usual inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle A, B \rangle = \operatorname{Re}\operatorname{tr}(AB^*) \qquad \text{for all} \quad A, B \in \mathbb{C}_{m\times n},$$

one may consider the dual transformation of a real linear operator on $\mathbf{C}_{m\times n}$. By the same technique used in the proof of Theorem 1, we have

THEOREM 1′. *Let $c,d\in\mathbf{R}_+^m\downarrow$. A real linear operator $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ will satisfy*

$$\|T(A)\|_d=\|A\|_c \qquad \textit{for all } A$$

if and only if its dual transformation T^* satisfies

$$T^*(\mathscr{S}(d))=\mathscr{S}(c).$$

We can compute as in the proof of Lemma 12 that the tangent space to $\mathscr{S}_1$ at E_{11} has dimension $2(n+m)-3$, whereas the tangent space to $\mathscr{S}_m$ at $\sum_{i=1}^m E_{ii}$ has dimension $m(2n-m)$. These two dimensions can never be equal if $(m,n)\neq(3,3)$. When $m=n=3$, we can show (by similar but more involved computations than in the proof of Lemma 8) that there are no $r>0$ and real linear T on $\mathbf{C}_{3\times 3}$ that satisfy $T(\mathscr{S}_1)=r\mathscr{S}_3$. Together with Lemma 7, this implies $T(c_1^{-1}\mathscr{S}_1)=d_1^{-1}\mathscr{S}_1$ if T is a real isometry from $(\mathbf{C}_{m\times n},\|\cdot\|_c)$ to $(\mathbf{C}_{m\times n},\|\cdot\|_d)$ and $c_1^{-1}\mathscr{S}_1\subset\mathscr{E}(c)$.

Similar to Theorems 3 and 4, we have

THEOREM 3′. *Let $c,d\in\mathbf{R}_+^m\downarrow$. Then the following conditions are equivalent:*

(a) *c and d are scalar multiples of each other.*

(b) *There exists a real isometry from $(\mathbf{C}_{m\times n},\|\cdot\|_c)$ to $(\mathbf{C}_{m\times n},\|\cdot\|_d)$.*

(c) *There exists a real linear operator $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ which maps $\mathscr{S}(d)$ onto $\mathscr{S}(c)$.*

THEOREM 4′. *Let $c=(c_1,\dots,c_m),d=(d_1,\dots,d_m)\in\mathbf{R}_+^m\downarrow$ be scalar multiples of each other, and $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ be real linear. Then the following conditions are equivalent:*

(a) *T is a real isometry from $(\mathbf{C}_{m\times n},\|\cdot\|_c)$ to $(\mathbf{C}_{m\times n},\|\cdot\|_d)$.*

(b) *$T(\mathscr{S}(d))=\mathscr{S}(c)$.*

(c) *There exist $U\in\mathscr{U}_m$ and $V\in\mathscr{U}_n$ where*

$$\mathscr{U}_k=\{A\in\mathbf{C}_{k\times k}:AA^*=I\}$$

and one of the following holds:

(i) $T(A)=UAV$ *for all* A;

(ii) $T(A)=U\bar{A}V$ *for all* A;

(iii) $m=n$ *and* $T(A)=UA^tV$ *for all* A; *or*

(iv) $m=n$ *and* $T(A)=UA^*V$ *for all* A.

Proofs of Theorems 3′ and 4′ are similar to those of Theorems 3 and 4. As there are no new ideas involved, we omit these lengthy proofs. We remark that the exceptional condition (c)(iii) of Theorem 6 does not occur here, and $\mathscr{S}_m$ is always connected. Particular cases of Theorem 4′ can be found in [10].

7. SOME REMARKS

(1) The proofs of the theorems in Sections 4 to 6 are lengthy. It would be nice to have shorter, simpler proofs for them.

(2) Suppose $c=d=(1,\dots,1)$. For $\mathbb{F}=\mathbb{C}$, it is known that if $T(\mathscr{S}_m)\subset\mathscr{S}_m$, then T satisfies condition (c) of Theorem 4 [14, 4]. For $\mathbb{F}=\mathbb{R}$ and $m=n$, it is also known that if T is nonsingular and $T(\mathscr{S}_m)\subset\mathscr{S}_m$, then condition (c) of Theorem 6 holds [16]. In general, it would be interesting to know how far one could weaken the condition $T(\mathscr{S}(d))=\mathscr{S}(c)$ in the theorems to get the same conclusions.

(3) The idea of characterizing linear operators via their duals has been used by Kovacs [9]. Grone and Marcus [6] used the characterization of the extreme points of the unit sphere to study the Ky Fan k-norm preservers. We used both of these ideas in this paper, as well as in an earlier paper [11] to study the preservers of the generalized numerical radius or hermitian matrices with fixed eigenvalues.

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